Exercise 26

The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

Solution

Take the derivative of $n(t)$ to get the rate of population growth per hour.

$$\begin{aligned}
\frac{dn}{dt} &= \frac{d}{dt}\left(\frac{a}{1 + be^{-0.7t}}\right) \\
&= \frac{\left[\frac{d}{dt}(a)\right](1 + be^{-0.7t}) - \left[\frac{d}{dt}(1 + be^{-0.7t})\right](a)}{(1 + be^{-0.7t})^2} \\
&= \frac{(0)(1 + be^{-0.7t}) - \left[(be^{-0.7t}) \cdot \frac{d}{dt}(-0.7t)\right](a)}{(1 + be^{-0.7t})^2} \\
&= \frac{-(be^{-0.7t}) \cdot (-0.7)(a)}{(1 + be^{-0.7t})^2} \\
&= \frac{0.7abe^{-0.7t}}{(1 + be^{-0.7t})^2}
\end{aligned}$$

Use the given information to construct a system of two equations for the two unknowns, a and b.

$$\begin{cases} n(t = 0) = \dfrac{a}{1 + be^{-0.7(0)}} = 20 \\[2ex] \dfrac{dn}{dt}(t = 0) = \dfrac{0.7abe^{-0.7(0)}}{(1 + be^{-0.7(0)})^2} = 12 \end{cases}$$

$$\begin{cases} \dfrac{a}{1 + b} = 20 \\[2ex] \dfrac{0.7ab}{(1 + b)^2} = 12 \end{cases}$$

Solve the first equation for $1/(1 + b)$ and b,

$$\frac{a}{1 + b} = 20 \quad \rightarrow \quad \frac{1}{1 + b} = \frac{20}{a} \quad \rightarrow \quad 1 + b = \frac{a}{20} \quad \rightarrow \quad b = \frac{a}{20} - 1, \tag{1}$$

and substitute the formulas into the second equation.

$$\frac{0.7ab}{(1+b)^2} = 12$$

$$0.7ab\left(\frac{1}{1+b}\right)^2 = 12$$

$$0.7a\left(\frac{a}{20} - 1\right)\left(\frac{20}{a}\right)^2 = 12$$

$$0.7a\left(\frac{20}{a} - \frac{20^2}{a^2}\right) = 12$$

$$14\left(1 - \frac{20}{a}\right) = 12$$

Solve for a.

$$1 - \frac{20}{a} = \frac{6}{7}$$

$$-\frac{20}{a} = -\frac{1}{7}$$

$$\frac{20}{a} = \frac{1}{7}$$

$$\frac{a}{20} = 7$$

$$a = 140$$

Substitute this back into equation (1) to determine b.

$$b = \frac{a}{20} - 1 = \frac{140}{20} - 1 = 7 - 1 = 6$$

As a result, the population function is

$$n(t) = \frac{140}{1 + 6e^{-0.7t}}.$$

Take the limit as $t \to \infty$ to find what happens to the yeast population in the long run.

$$\lim_{t\to\infty} n(t) = \lim_{t\to\infty} \frac{140}{1 + 6e^{-0.7t}} = \frac{140}{1 + 6(0)} = 140$$

This indicates that the yeast population tends towards 140 cells over a very long time.